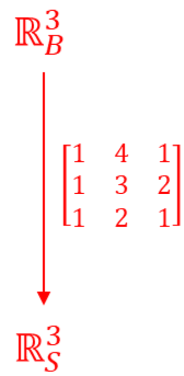
Name ____ Test 2, Spring 2021

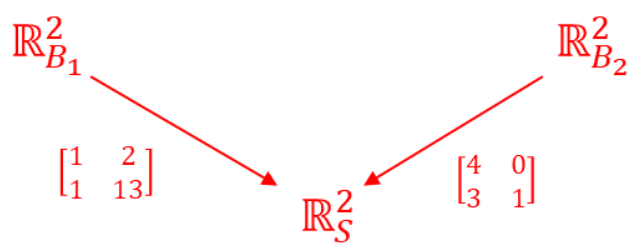
1) Given the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and $\vec{x}_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_S$, find a formula for $[\vec{x}]_B$. (10 points)

$$\begin{bmatrix} 1 & 4 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



2) Given the bases $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 13 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, find a formula for the change of basis matrix that converts vectors from basis B_1 into vectors from basis B_2. (10 points)

$$[I]_{B_1}^{B_2} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 13 \end{bmatrix}$$



3) Find the determinant of the matrix below. (15 points)

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 3 & 4 & 0 & 2 \end{bmatrix}$$

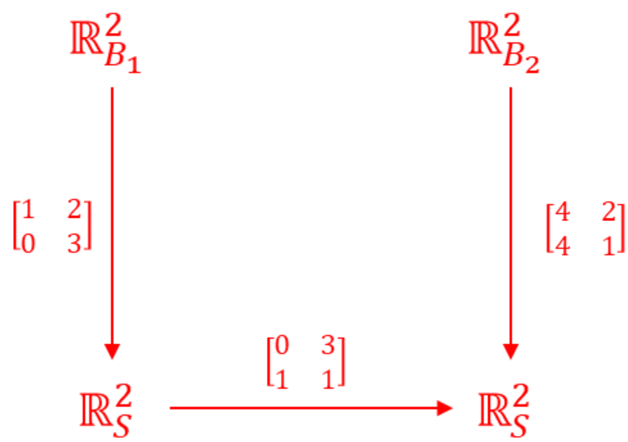
$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 3 & 4 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= (6 - 20) - 2(2 - 15) + 3(4 - 9) = -14 + 26 - 15 = -3$$

4) Given the linear transformation $T: \mathbb{R}^2_S \to \mathbb{R}^2_S$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S\right) = \begin{bmatrix} 3x_2 \\ x_1 + x_2 \end{bmatrix}_S$ and the bases below, find a formula for $\left[T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{B_1}\right)\right]_{B_2}$. (10 points)

$$B_1 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$
$$B_2 = \left\{\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

$$\left[T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{B_1}\right)\right]_{B_2} = \begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



5) Answer the following questions. (3 points each)

A) Let A be a 3×3 matrix and assume that it has rank 2. How many solutions does $A\vec{x} = \vec{0}$ have?

∞

B) Let A be a 3×4 matrix and assume that the corresponding linear transformation T is not onto. What is the minimum dimension of the null space of A?

2

C) Let A be a 3×7 matrix. Assume that the dimension of the row space is 3. What is the dimension of the column space?

3

D) Consider a system of 5 equations in 3 variables. Assume there are infinitely many solutions. If A is the matrix representing this system, what are the possible values for the rank of A?

0, 1, or 2

E) Let A be a 6×6 matrix and T the corresponding linear transformation. If $\dim(\ker(T)) = 2$, what is the rank of A?

4

6) Find the null space of the matrix below. (10 points)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

x_3 is free
$x_2 = -4x_3$
$x_1 = -3x_3$

$$NS = span\left(\left\{\begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}\right\}\right)$$

7) Find the product below. (5 points)

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 & 4 & 3 \\ 2 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 2 \\ 6 & 5 & 2 & 7 & 9 \end{bmatrix}$$

Use the fact that they're elementary matrices!

$$\begin{bmatrix} 5 & 6 & 8 & 14 & 13 \\ 2 & 2 & 3 & 4 & 5 \\ 1 & 4 & 4 & 7 & 7 \\ 0 & 2 & 1 & 3 & 2 \\ 12 & 10 & 4 & 14 & 18 \end{bmatrix}$$

You may be interested in the information below for the questions on this page.

$$\begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -\frac{4}{3} & \frac{7}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8) Are the vectors linearly dependent or linearly independent? Why? (5 points)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \right\}$$

No, notice that when row reduced there are columns without pivots.

9) Can $\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$ be written as a unique linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$? Why or why not? (5 points)

No.

It can be written as a linear combination, but notice the free variable in the augmented matrix. It is not unique.

10) Find a basis for the vector space below. (5 points)

$$span\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \right\} \right)$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \right\}$$

11) Given the information below regarding the linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $S: \mathbb{R}^3 \to \mathbb{R}^2$, find the diagram that illustrates them as well as $T \circ S$. (10 points)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 3x_2 \\ x_1 + x_2 \end{bmatrix} \qquad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 4x_1 + x_3 \end{bmatrix}$$

$$\mathbb{R}^3 \xrightarrow{\begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \mathbb{R}^2 \xrightarrow{\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}} \mathbb{R}^3$$